A COMPACTNESS THEOREM FOR RIEMANNIAN MANIFOLDS WITH BOUNDARY AND APPLICATIONS

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Abstract. In this paper we prove weak $L^{1,p}$ (and thus C^α) compactness for the class of uniformly mean-convex Riemannian n-manifolds with boundary satisfying bounds on curvature quantities, diameter, and $(n-1)$-volume of the boundary. We obtain two stability theorems from the compactness result. The first theorem applies to 3-manifolds (contained in the aforementioned class) that have Ricci curvature close to 0 and whose boundaries are Gromov-Hausdorff close to a fixed metric on S^2 with positive curvature. Such manifolds are C^α close to the region enclosed by a Weyl embedding of the fixed metric into $\mathbb{R}^3$. The second theorem shows that if a 3-manifold with $\chi(\partial M) = 2$ has Ricci curvature close to 0 (resp. -2, 2) and mean curvature close to 2 (resp. $2\sqrt{2}$, 0), then M is C^α close to a metric ball in the space form of constant curvature 0 (resp -1, 1).

1. Introduction

Cheeger-Gromov compactness ([12], [8]) asserts that the class of Riemannian manifolds (of fixed dimension) satisfying $|\sec(M)| \leq K$, $\operatorname{diam}(M) \leq D$, and $\operatorname{vol}(M) \geq v_0$ is precompact in the $C^{1,\alpha}$ topology, any $0 < \alpha < 1$. Here $\sec(M)$, $\operatorname{diam}(M)$ and $\operatorname{vol}(M)$ respectively refer to the sectional curvature, diameter and volume. It is natural to ask if a result like this is true for certain classes of manifolds with boundary. Kodani ([17]) has proven an analogue of Gromov's compactness theorem for manifolds with boundary. To describe the result, first consider the class of Riemannian n-manifolds with boundary satisfying the conditions

$$\operatorname{vol}(M) \geq V,\ \operatorname{diam}(M) \leq D,$$
$$|\sec(M)| \leq K,\ \lambda_- \leq II \leq \lambda_+.$$

Here II is the second fundamental form of ∂M in M. If λ_- is allowed to be arbitrary, then this class is not even precompact in the C^0 topology. However, Kodani shows that there exists $\lambda^* < 0$, depending upon n, λ_+, K, D, V, and $\operatorname{diam}(\partial M)$, so that if $\lambda_- > \lambda^*$, then any sequence in the class subconverges in the Lipschitz topology to a limiting C^0 Riemannian metric. The precise control required of λ_- is a definite restriction on the applicability of the theorem. Moreover, the regularity of the limiting metric is not optimal. Since the mean curvature involves one derivative of the metric, one expects to gain control of the first derivatives of g and therefore obtain, for instance, an $L^{1,p}$ limit.

Extending techniques introduced by Anderson ([4]), it is shown in [3] that the class of Riemannian n-manifolds with boundary satisfying

$$|\operatorname{ric}(M)| \leq K, \quad |\operatorname{ric}(\partial M)| \leq K$$
$$\operatorname{inj}(M) \geq i_0, \quad \operatorname{inj}(\partial M) \geq i_0, \quad i_b(M) \geq i_0$$
$$\operatorname{diam}(M) \leq D, \qquad |H|_{Lip} \leq H_0$$

is precompact in the weak C^2_* topology. C^2_* is the Zygmund space intermediate between $C^{1,\alpha}$ and C^2, i_b is the boundary injectivity radius defined in section 2 and $|\cdot|_{Lip}$ is the Lipschitz norm. Weak

C^2_* precompactness means that any sequence subconverges in the $C^{1,\alpha}$ topology (any $0 < \alpha < 1$) to a C^2_* limit. The C^2_* limit is necessary for the applications in [3], therefore Lipschitz control of H is natural in this context. We note that if one strengthens the control of H in our Theorem 1.1 from pointwise to Lipschitz, then the techniques in [3] allow us to obtain weak C^2_* convergence as well. The details of this are minor however and we will not discuss them here (see also the remarks below Theorem 1.1).

Let us assume throughout that all manifolds being considered are connected. Write $\mathcal{M}$ for the class of compact Riemannian n-manifolds with connected boundary satisfying

$$|\sec(M)| \leq K,\ |\sec(\partial M)| \leq K$$
$$0 < 1/H_0 < H < H_0$$
$$\mathrm{diam}(M) \leq D,\ \mathrm{area}(\partial M) \geq A_0.$$

Theorem 1.1. *$\mathcal{M}$ is precompact in the C^α and weak $L^{1,p}$ topologies, for any $0 < \alpha < 1$ and any $p < \infty$. Consequently $\mathcal{M}$ has only finitely many diffeomorphism types.*

The sectional (and thus Ricci) curvature bounds give $L^{2,p}$ control of the metric in appropriate charts (e.g. harmonic coordinate charts) in the interior of M. Similar analysis applied to the intrinsic boundary metric ∂M gives $L^{2,p}$ control of $g_{\partial M}$ in charts of uniform size on ∂M. This information can be used to gain $L^{2,p}$ control of the tangential components of the metric in a neighborhood of the boundary (see the proof of Theorem 1.1). The nontangential components of the metric however are controlled by the mean curvature. Here one only obtains $L^{1,p}$ control of these components in a neighborhood of the boundary, so that the regularity part of Theorem 1.1 is limited by the pointwise control of H. Therefore we could strengthen the convergence result by controlling H in a stronger norm. If we assume for instance that H is controlled in an appropriate trace space $L^{2-1/p,p}$ (see [7] for an example of an 'appropriate trace space'), then we obtain weak $L^{2,p}$ convergence in the statement of Theorem 1.1. We could similarly assume that H is bounded in the C^α or Lipschitz topology (as in [3]) and respectively obtain a weak $C^{1,\alpha}$ or weak C^2_* convergence result. Since the proofs of these results are similar to the proof of Theorem 1.1 (and because it is more natural from a geometric standpoint to work with a pointwise bound on H) we will leave the details to the interested reader.

We will use Theorem 1.1 to prove two 'geometric stability theorems' regarding 3-manifolds with boundary. Write (Σ, h_Σ) for a smooth, closed, oriented surface with Gauss curvature $K_\Sigma > 0$. From the solution of the Weyl problem (cf. [13]), there exists a smooth isometric embedding $\Sigma \to \mathbb{R}^3$ whose image is unique up to rigid motion. Choose such an immersion i and write N for the convex solid region bounded by $i(\Sigma)$. Then $N \subset \mathbb{R}^3$ is a smooth, flat manifold with boundary $\partial N = \Sigma$.

Theorem 1.2. *Suppose (M, g) is a compact, oriented, simply connected Riemannian 3-manifold with connected boundary. Write h for the induced metric on ∂M and write K for the Gauss curvature of h. Suppose that $H > 0$ and $K > 0$. To every $\epsilon > 0$ there exists a number $\delta = \delta(\epsilon, \sup H, \sup K, \inf K, \alpha)$ so that if*

$$(h, h_\Sigma)_{GH} < \delta, \quad |\,ric(g)| < \delta$$

then there exists a diffeomorphism $f : N \to M$, and

$$||f^* g - g_{Euc}||_{C^\alpha} \leq \epsilon,$$

where g_{Euc} is the standard Euclidean metric on N and $(\cdot, \cdot)_{GH}$ is the Gromov-Hausdorff distance.

Using the same techniques we can obtain a somewhat different result in the special case that N is a ball. Write B for the unit ball in $\mathbb{R}^3$.

Corollary 1.1. *Suppose (M,g) is a compact oriented Riemannian 3-manifold with connected boundary and that $H>0$. To every $\epsilon>0$ there exists $\delta=\delta(\epsilon,\sup H,\inf H,\alpha)$ so that if*

$$|K-1|<\delta,\quad |\,ric(g)|<\delta$$

then there exists a diffeomorphism $f:B\to M$, and

$$||f^*g-g_{Euc}||_{C^\alpha}\leq\epsilon.$$

Theorem 1.2 can be viewed as a generalization of the well-known rigidity theorem of Cohn-Vossen ([10], [13]). Cohn-Vossen's theorem, part of the solution of the Weyl problem mentioned above, states that an analytic immersion

$$i:\Sigma\to\mathbb{R}^3$$

of a closed surface Σ with Gauss curvature $K>0$ has a unique image modulo a rigid motion of $\mathbb{R}^3$. Pogorelov ([21]) removed the restriction on the regularity of the immersion. The result can be restated, via the developing map, as a theorem about flat 3-manifolds with boundary. Thus if (M_1,g_1) and (M_2,g_2) are compact, simply connected, flat 3-manifolds with isometric boundaries that have positive Gauss curvature, then M_1 is diffeomorphic to M_2 and the metrics g_1, g_2 are in the same isometry class. Such manifolds are therefore 'geometrically rigid.' Theorem 1.2 is a natural generalization of Cohn-Vossen's rigidity theorem in this context.

Let us provide another application of Theorem 1.1, motivated by Hopf's rigidity theorem ([15]). Hopf's theorem states that the image of a C^3 isometric immersion

$$i:S^2\to\mathbb{R}^3$$

of a metric on S^2 with constant mean curvature is a (Euclidean) sphere. Essentially the same proof shows that the image of a C^3 isometric immersion

$$i:S^2\to\mathbb{H}^3$$

of a metric on S^2 into hyperbolic space is the boundary of a metric ball, provided $H>2$. Here H is the trace of the second fundamental form with respect to the outward normal, so that in our notation every distance sphere in $\mathbb{H}^3$ has mean curvature $H>2$. Almgren ([1]) has shown (making use of Hopf's proof) that any analytic minimal immersion of S^2 into S^3 is congruent to the equator. To each of these rigidity theorems we associate a geometric stability theorem.

Theorem 1.3. *Let (M,g) be a compact, oriented Riemannian 3-manifold with connected boundary and suppose that $\chi(\partial M)=2$. To every $\epsilon>0$ there exists $\delta=\delta(\sup|S|,diam(M),\alpha)$ such that*

i) if

$$|\,ric(M)|\leq\delta,\quad |H-2|\leq\delta$$

then there exists a diffeomorphism $f:B\to M$ and

$$||f^*g-g_{Euc}||_{C^\alpha}\leq\epsilon.$$

ii) if

$$|\,ric(g)+2g|\leq\delta,\quad |H-2\sqrt{2}|\leq\delta$$

then there exists a diffeomorphism $f:B_H\to M$ and

$$||f^*g-g_{-1}||_{C^\alpha}\leq\epsilon,$$

where B_H is a metric ball in hyperbolic space with Gauss curvature of the boundary equal to 1*, and g_{-1} is the standard metric of curvature* -1*.*

iii) if

$$|\,ric(g) - 2g| \leq \delta, \quad |H| \leq \delta$$

then there exists a diffeomorphism $f : S^3_+ \to M$ and

$$||f^* g - g_{+1}||_{C^\alpha} \leq \epsilon,$$

where S^3_+ is the upper hemisphere in $S^3 \subset \mathbb{R}^4$ and g_{+1} is the standard metric of curvature $+1$.

In section 2 we provide definitions and well-known background results necessary for the proofs to follow. The proof of Theorem 1.1 is contained in section 3, the proofs of Theorem 1.2 and Corollary 1.1 are contained in section 4, and the proof of Theorem 1.3 is contained in section 5.

Acknowledgement. The author would like to thank his thesis adviser Michael Anderson for suggesting the topic and for many valuable comments and conversations.

2. Background Results

Let us begin with a few observations about the elements of $\mathcal{M}$. Write $t(x) = \mathrm{dist}(x, \partial M)$ and note that t is smooth off of the cut locus of ∂M. Write

$$\nu = -\mathrm{grad}\, t$$

so that ν is the outward normal of ∂M. Write $g(S(X), Y)$ for the second fundamental form, thus $S(X) = \nabla_X \nu$. Here ∇ is the usual covariant derivative. The mean curvature H of ∂M is the trace $H := \mathrm{tr}\, S$. If e_i is a basis for $T_p \partial M$ such that $S(e_i) = \lambda_i e_i$, then the Gauss equation reads

$$\lambda_i \lambda_j = \sec_{\partial M}(e_i, e_j) - \sec_M(e_i, e_j).$$

This implies that each pair $\lambda_i \lambda_j$ is uniformly bounded. Together with the fact that $H = \sum \lambda_i$ is uniformly controlled we see that each λ_i is uniformly controlled as well (in fact $|\lambda_i| \leq \max\{H_0 + 2(n-2)K, 1\}$).

A consequence of the uniform bound on S is that the elements of $\mathcal{M}$ have a uniform upper bound on the intrinsic diameter of the boundary (see [23, Thm 1.1]). Thus we will assume without loss of generality that $\mathrm{diam}(\partial M) \leq D$.

Write $\mathrm{foc}(\partial M)$ for the focal locus distance of ∂M. Let us recall a basic result from comparison geometry.

Lemma 2.1. *Suppose λ and K are any real numbers. Let t_0 be the smallest positive solution to*

$$\begin{aligned} \cot \sqrt{K} t &= \frac{\lambda}{\sqrt{K}} \qquad \textit{if } K > 0 \\ t &= \frac{1}{\lambda} \qquad \textit{if } K = 0 \\ \coth \sqrt{-K} t &= \frac{\lambda}{\sqrt{-K}} \qquad \textit{if } K < 0 \end{aligned}$$

(a) If $S \leq \lambda$ and $\sec(M) \leq K$, then $foc(\partial M) \geq t_0$.
(b) If $S \geq \lambda$ and $K \geq \sec(M)$, then $foc(\partial M) \leq t_0$.

Write $\mathrm{cut}(\partial M)$ for the cut locus distance of ∂M. Write $\nu(\partial M)$ for the normal bundle of ∂M and define the normal exponential map

$$\exp_\nu : \nu(\partial M) \to M.$$

Define the boundary injectivity radius i_b to be the largest t such that $\exp_\nu$ is a diffeomorphism on $\partial M \times [0, t)$. It is a standard fact that

$$i_b \geq \min\{\mathrm{foc}(\partial M), \mathrm{cut}(\partial M)\}.$$

Let us show, using the proof of [2, Lemma 2.4], that any $(M, g) \in \mathcal{M}$ has i_b uniformly bounded below.

Lemma 2.2. *Suppose* $(M, g) \in \mathcal{M}$. *Then*

$$i_b(M) \geq \min\{t_0, \frac{2}{(n-1)H_0 K}\}$$

for t_0 *as defined in Lemma 2.1. If in addition* $ric(g) \geq 0$, *then* $i_b(M) \geq t_0$.

Proof. Since $\mathrm{foc}(\partial M) \geq t_0$, it is enough to show that if $\mathrm{cut}(\partial M) < t_0$, then

$$\mathrm{cut}(\partial M) \geq \frac{2}{(n-1)H_0 K}. \tag{2.1}$$

Therefore suppose that $\mathrm{cut}(\partial M) < t_0$. Choose an arclength parametrized geodesic γ realizing the cut locus distance. Thus

$$\gamma : [0, l] \to M$$

is a minimizing geodesic from $p \in \partial M$ to $q \in \partial M$ that is orthogonal to ∂M at the endpoints and such that $\mathrm{Image}(\gamma) \cap \partial M = \{p, q\}$. Consider the index form

$$I(V, W) = \int_0^l g(\nabla_{\gamma'} V, \nabla_{\gamma'} W) - g(R(\gamma', V)W, \gamma')\, dt - g(S(V), W)|_0^l, \tag{2.2}$$

where V and W are vector fields along γ and orthogonal to γ'. Since there are no focal points along γ it follows that

$$I(V, V) \geq 0$$

for any such V. Therefore choose an orthonormal basis e_i for T_pM and define V_i to be the parallel translation of e_i along γ. Then

$$\begin{aligned} 0 \leq \sum_i I(V_i, V_i) &= -\int_0^l \mathrm{ric}(\gamma', \gamma')\, dt - (H(p) + H(q)) \\ &\leq (n-1)lK - 2/H_0, \end{aligned}$$

establishing the first assertion. If in addition $\mathrm{ric}(g) \geq 0$, then

$$0 \leq -2/H_0$$

which contradicts the fact that $\mathrm{cut}(\partial M) < t_0$. $\square$

2.1. **Convergence results.** A sequence (M_i, g_i) of Riemannian manifolds (with or without boundary) converges to (M, g) in the $L^{k,p}$ topology, $k \geq 1$, if $g \in L^{k,p}$ and there exist diffeomorphisms $F_i : M \to M_i$ so that $F_i^* g_i \to g$ in the $L^{k,p}$ topology on M. $L^{k,p}(M)$ is the usual Sobolev space of functions (or tensors) with k weak derivatives in L^p. To be somewhat more precise about the definition of convergence, we require that (M, g) has an $L^{k+1,p}$ atlas of charts in which $F_i^* g_i \to g$ in $L^{k,p}_{\text{loc}}$ in each chart. Similar definitions hold for convergence in other function spaces. For an introduction to the convergence theory of Riemannian manifolds see for instance [12], [20], [9],[14], [22].

It is useful to discuss convergence theory in terms that only refer to the local geometry of M. For this purpose we use terminology first introduced by Anderson in [4]. Given $p \in M \backslash \partial M$ and a number $Q > 1$, define the $L^{k,p}$ harmonic radius at p, denoted $r_h^Q(p) := r_h(p)$, to be the largest number $r < \text{dist}(p, \partial M)$ satisfying the following conditions. There exists a harmonic coordinate system $\{\phi_i\}$ (i.e. $\Delta_g \phi_i = 0$) centered at p and containing the geodesic r-ball $B_r(p)$ on which there holds, for each multi-index σ with $|\sigma| = l \leq k$,

$$Q^{-1}\delta_{ij} \leq g_{ij} \leq Q\delta_{ij}, \tag{2.3}$$

$$r^{l-n/p}||\partial^\sigma g_{ij}||_{L^p} \leq Q \tag{2.4}$$

where δ_{ij} is the standard Euclidean metric. The conditions $(2.3) - (2.4)$ are invariant under simultaneous rescalings of the metric and the coordinates, so that r_h scales like a distance function, i.e. $r_h(\lambda^2 g) = \lambda r_h(g)$. Throughout the discussion we will assume that Q is fixed, say $Q = 3/2$, so in particular there is no loss of clarity when supressing the dependence of $r_h(p)$ on Q. If $q \in \partial M$, define $r_h^Q(q) := r_h(q)$ to be the largest r such that $r < r_h^{\partial M}(q)$ and such that there exists a harmonic coordinate system centered at q, containing the geodesic r-balls $B_q(r)$ and $B_q^{\partial M}(r)$, and satisfying equations (2.3)-(2.4). Here $r_h^{\partial M}(q)$ refers to the harmonic radius of q in the manifold ∂M, and $B_q^{\partial M}(r)$ is defined to be the ball of radius r in ∂M. A harmonic coordinate system at $q \in \partial M$ is defined to be a system of coordinates $\{\phi_i\}$ so that $\phi_1, \ldots, \phi_{n-1}$ form a harmonic coordinate system at q in ∂M, $\phi_n = 0$ on ∂M and each ϕ_i is harmonic on M. We then define the harmonic radius of M, $r_h(M)$, to be the largest r so that for each p in M, either $r_h(p) > r$ or there exists $q \in \partial M$ with $r_h(q) > 2r$ and $p \in B_r(q)$.

It is clear how to extend the definition of $L^{k,p}$ harmonic radius to other function spaces. We could also consider for instance the $C^{k,\alpha}$ harmonic radius, $k \geq 0$, $0 < \alpha < 1$. By the Sobolev embedding theorem, if $p > n$ then the $L^{k,p}$ harmonic radius controls the $C^{k-1,\alpha}$ harmonic radius, $\alpha \leq n/p$. The harmonic radius of a manifold with boundary was previously defined and studied in [3], where it is shown that (for instance) if (M, g) is a Riemannian manifold with $g \in L^{1,p}$, then M admits an $L^{2,p}$ atlas of harmonic or boundary harmonic coordinate charts in which $g \in L^{1,p}$ (see also [11]).

In order to focus our attention on the local geometry of M near the boundary, we will also define the boundary harmonic radius $r_{bh} := \inf_{q \in \partial M}\{r_h(q)\}$. The boundary harmonic radius retains all of the important properties of the harmonic radius. For instance, it is continuous with respect to the $L^{k,p}$ topology, $k \geq 1$.

Lemma 2.3. *Suppose that $(M_i, g_i) \to (M, g_\infty)$ in the $L^{k,p}$ topology, $k \geq 1$. Then*

$$\lim_i r_{bh}(g_i) = r_{bh}(g_\infty).$$

The corresponding result for the $L^{1,p}$ harmonic radius of a complete manifold without boundary is done in [6], and the continuity of the C^2_* boundary harmonic radius (under C^2_* convergence) is done in [3]. The proof for the $L^{k,p}$ boundary harmonic radius is nearly identical to these cases, thus

we will not describe it here.

The same method used in the proof of Lemma 2.3 also implies that for a fixed manifold (M, g) the function $r_{bh} : \partial M \to \mathbb{R}_+$ is continuous.

The link between the harmonic radius and convergence is established by the following theorem (See for instance [16] and [20, Theorem 72]. The proofs there are about $C^{k,\alpha}$ convergence of manifolds without boundary. However, the Banach-Alaoglu theorem allows the result to be extended to weak $L^{k,p}$ convergence, and we may extend the result to manifolds with boundary by simply including boundary harmonic coordinate charts in the analysis.)

Theorem 2.1. *Suppose (M_i, g_i) is a sequence of Riemannian manifolds with boundary such that (for some $k \geq 1$) the $L^{k,p}$ harmonic radius $r_h(g_i) \geq r_0$ and diam$(M_i) \leq D$. Then there exists a smooth manifold with boundary (M, g), $g \in L^{k,p}(M)$, so that (M_i, g_i) subconverges in the weak $L^{k,p}$ topology to (M, g). In particular, the harmonic coordinate charts on M_i subconverge weakly in $L^{k+1,p}$ to harmonic coordinates on M.*

In the statement of Theorem 2.1 the diameter bound is used to obtain a uniform upper bound on the number of coordinate charts needed to cover M_i. We could also remove the diameter bound and consider pointed convergence. Given a sequence of points $p_i \in M_i$ we say that (M_i, g_i, p_i) converges to (M, g, p) in the $L^{k,p}$ topology if there exist real numbers $r_k < s_k$, $r_k \to \infty$, and compact sets $U_i \subset M_i$, $V_i \subset M$ so that

$$B_{r_i}(p_i) \subset U_i \subset B_{s_i}(p_i), \quad B_{r_i}(p) \subset V_i \subset B_{s_i}(p)$$

and diffeomorphisms

$$F_i : V_i \to U_i, \quad F_i : V_i \cap \partial M \to U_i \cap \partial M_i$$

so that $F_i^* g_i \to g$ in $L^{k,p}_{\text{loc}}$ and $F_i^{-1}(p_i) \to p$. A similar definition of pointed convergence could also be used to formulate a local version of Theorem 2.1.

We will make use of the following result.

Theorem 2.2. *Let (M, g) be a compact Riemannian n-manifold with boundary with $|\sec(M)| \leq K$. Suppose that for each $x \in M \backslash \partial M$ and each $r <$ dist$(x, \partial M)$ there holds*

$$vol(B_r(x)) \geq v_0 r^n.$$

Then for any $p < \infty$ there exists a constant $r_0 = r_0(n, K, v_0)$ so that the $L^{2,p}$ harmonic radius $r_h(q)$ satisfies

$$r_h(x) \geq r_0 \, dist(x, \partial M).$$

In the case where $\partial M = \emptyset$, volume comparison implies that it is sufficient to assume that $\text{vol}(M) \geq v_0$ and $\text{diam}(M) \leq D$. In this case Theorem 2.2 and Theorem 2.1 together imply the usual statement of Cheeger-Gromov compactness.

Proof of Theorem 2.2. We will outline the proof of this well-known result. Arguing by contradiction, suppose there were a sequence $(M_i, \tilde{g}_i, x_i)$ satisfying the hypothesis of Theorem 2.2 but with

$$\frac{\tilde{r}_i(x_i)}{\text{dist}(x_i, \partial M_i)} \to 0, \tag{2.5}$$

where $\tilde{r}_i(x_i) := \tilde{r}_i := r_h(x_i)$. Consider then the rescaled sequence $(M_i, \frac{1}{\tilde{r}_i^2}\tilde{g}_i) = (M_i, g_i)$. The scaling behavior of the harmonic radius implies that (calculated with respect to g_i) $r_h(x_i) = 1$. Since

equation (2.5) is scale-invariant it follows that

$$\mathrm{dist}_{g_i}(x_i, \partial M_i) \to \infty.$$

Thus Theorem 2.1 implies that (M_i, g_i, x_i) converges in the pointed weak $L^{2,p}$ topology to a complete Riemannian manifold (M_∞, g_∞, x).

As $|\sec(g_i)| \le \tilde{r}_i^2 K$ we see that $|\sec(g_i)| \to 0$. In particular, $|\mathrm{ric}(g_i)| \to 0$ in C^0. This allows one to improve the convergence from weak $L^{2,p}$ to (strong) $L^{2,p}$ (see [4]). Then Lemma 2.3 implies that $r_h(g_\infty) = 1$. However, the limit (M_∞, g_∞) is a complete, flat, C^∞ Riemannian manifold and is therefore isometric to a quotient of $\mathbb{R}^n$. The volume growth condition implies that (for all r)

$$\mathrm{vol}(B(x,r)) \ge v_0 r^n.$$

This implies that $(M_\infty, g_\infty) = (\mathbb{R}^n, g_{Euc})$, in contradiction to the fact that $r_h(g_\infty) = 1$. □

3. The Precompactness Theorem

The proof of Theorem 1.1 proceeds in two basic steps. The first is to control from below the Euclidean volume growth of any point p in the interior of M. The second is to show that the $L^{1,p}$ boundary harmonic radius r_{bh} is bounded from below. These two facts together with Theorem 2.2 imply that the harmonic radius $r_h(g)$ is uniformly bounded below, so that Theorem 2.1 establishes the result.

Let us begin by controlling the volume of small cylinders in M with base $B \subset \partial M$. Therefore define, for $0 \le t_1 \le t_2 < i_b/2$,

$$C(B, t_1, t_2) := \{\exp_\nu(q, t) : t_1 \le t \le t_2, \, q \in B\}.$$

Lemma 3.1. *Suppose $(M, g) \in \mathcal{M}$ and choose $q \in \partial M$. Suppose there exists $s > 0$ so that $B_q^{\partial M}(s) \subset B$. Then there exists a constant a_0, depending only upon $\mathcal{M}$, so that*

$$vol(C(B, t_1, t_2)) \ge a_0 s^{n-1}(t_2 - t_1).$$

Proof. First note that volume comparison (in ∂M) implies that

$$\mathrm{area}(B) \ge c s^{n-1} \tag{3.1}$$

for some c that only depends on n, $\mathrm{diam}(\partial M)$, and $\mathrm{ric}(\partial M)$. Let B_r be the level set

$$\{\exp_\nu(q, r) : q \in B_q^{\partial M}(s)\}.$$

Since $\mathrm{vol}(C(B, t_1, t_2)) \ge \int_{t_1}^{t_2} \mathrm{area}(B_r)\, dr$, it suffices to show that $\mathrm{area}(B_r)$ is uniformly controlled in terms of $\mathcal{M}$ and $\mathrm{area}(B)$. We note that if another constant c is chosen so that the inequality

$$\frac{1}{K(i_b - t)^2} > \frac{c^2}{(1-c)},$$

holds for all $0 < t < i_b/2$, then

$$H(t) \le \frac{n-1}{c(i_b - t)}. \tag{3.2}$$

This estimate is proved for instance in Lemma 3.2.2 of [3] (with a specific choice of c).

Write $A(r) := \mathrm{area}(B_r)$ and write $A_0 := \mathrm{area}(B)$. Consider the first variation of area

$$A'(r) = -\int_{B_r} H d\mu_r$$

where $d\mu_r$ is the volume form on B_r. Together with (3.2) this gives a differential inequality for the area

$$A'(r) \geq -\int_{B_r} \frac{(n-1)}{c(i_b - r)} d\mu_r = -\frac{(n-1)}{c(i_b - r)} A(r) \tag{3.3}$$

This implies that

$$A(r) \geq A_0 (i_b)^{-\frac{n-1}{c}} (i_b - r)^{\frac{n-1}{c}}. \tag{3.4}$$

Since i_b only depends on $\mathcal{M}$ (Lemma 2.2), $A(r)$ only depends on $\mathcal{M}$ and area(B). $\square$

Proposition 3.1. *There exists a constant $v_0 = v_0(\mathcal{M})$ so that for each $x \in (M, g) \in \mathcal{M}$ and each $r <$ dist$(x, \partial M)$,*

$$vol(B_r(x)) \geq v_0 r^n.$$

Proof. Put $r_x = \text{dist}(x, \partial M)$. Choose an arclength parametrized minimizing geodesic γ from x to $q \in \partial M$. Consider the unique point y in the image of γ satisfying $r_y = \min(r_x, i_b/2)$. Write Σ_t for the level set of constant t from ∂M and suppose that $t \leq i_b/2$. Hessian comparison implies that there exists a constant $C = C(\mathcal{M})$ so that if $\text{dist}_{\partial M}(p, q) \leq \epsilon$, then $\text{dist}_{\Sigma_t}(\exp_\nu(p, t), \exp_\nu(q, t)) \leq C\epsilon$. Put

$$B := B^{\partial M}_{\frac{r_y}{4C}}(q), \tag{3.5}$$

i.e. the $(r_y/4C)$-ball about q in ∂M. From the triangle inequality we see that

$$C(B, 3/4r_y, r_y) \subset B_{r_y}(y) \subset B_{r_x}(x), \tag{3.6}$$

where $C(B, 3/4r_y, r_y)$ is the cylinder defined in Lemma 3.1. Then by Lemma 3.1 there exists $a_0 = a_0(\mathcal{M})$ so that

$$\text{vol}(B_{r_x}(x)) \geq \text{vol}(C(B, 3/4r_y, r_y)) \geq a_0 r_y^n. \tag{3.7}$$

Thus either

$$\text{vol}(B_{r_x}(x)) \geq a_0 r_x^n$$

or

$$\text{vol}(B_{r_x}(x)) \geq \frac{a_0 i_b^n}{2^n}.$$

In either case, volume comparison applied to the ball $B_{r_x}(x)$ establishes the desired result. $\square$

It remains to show that the boundary harmonic radius is bounded from below.

Proposition 3.2. *For any $p > 2n$ there exists $r_0 > 0$, depending only upon $\mathcal{M}$, so that for any $(M, g) \in \mathcal{M}$, the $L^{1,p}$ boundary harmonic radius $r_{bh} \geq r_0$.*

Proof. We proceed by contradiction. If the conclusion were false, then there exists a sequence $(M_i, \tilde{g}_i) \in \mathcal{M}$ so that $\tilde{r}_i = r_{bh}(g_i) \to 0$. Choose points $p_i \in \partial M_i$ satisfying $r_{bh}(p_i) = \tilde{r}_i$ and consider the normalized sequence

$$(M_i, g_i, p_i) := (M_i, \frac{1}{\tilde{r}_i^2} \tilde{g}_i, p_i).$$

Note that $r_i = r_{bh}(g_i) = 1$ and so Theorems 2.1 and 2.2 imply that the sequence (M_i, g_i, p_i) converges weakly in $L^{1,p}$ to a limit (M_∞, g_∞, p). Write $\tilde{h}_i$ and h_i for the metrics on ∂M_i induced respectively from $\tilde{g}_i$ and g_i. Due to the scaling properties of the various quantities we see that

$$
\begin{aligned}
|\sec(h_i)| &\leq \tilde{r}_i^2 K \to 0 \\
H_i &\leq \tilde{r}_i H_0 \to 0 \\
|\sec(g_i)| &\leq \tilde{r}_i^2 K \to 0 \\
i_b(g_i) &= \frac{1}{\tilde{r}_i} i_b(\tilde{g}_i) \to \infty.
\end{aligned}
$$

The metrics $\tilde{h}_i$ have bounded curvature, diameter and volume so that passing to a subsequence if necessary we can assume (as in the proof of Lemma 2.2) that $(\partial M_i, h_i, p_i)$ converges to $(\mathbb{R}^{n-1}, g_{Euc}, 0)$ in $C^{1,\alpha}$ and $L^{2,p}$. The limit (M_∞, g_∞) satisfies (in the $L^{1,p}$ sense)

$$\operatorname{ric}(h_\infty) = 0 \tag{3.8}$$

$$H_\infty = 0 \tag{3.9}$$

$$\operatorname{ric}(g_\infty) = 0. \tag{3.10}$$

As noted in [3], when expressed in boundary harmonic coordinates on (M_∞, g_∞) one obtains the system of equations (writing $g := g_\infty$ for the moment)

$$\Delta g^{in} = B^{in}(g, \partial g) - 2\operatorname{ric}(g)^{in} = B^{in}(g, \partial g) \tag{3.11}$$

$$\partial_t g^{nn} = -2(n-1)Hg^{nn} = 0 \tag{3.12}$$

$$\partial_t g^{\alpha n} = -(n-1)Hg^{\alpha n} + \frac{1}{2\sqrt{g^{nn}}} g^{\alpha k}\partial_k g^{nn} = \frac{1}{2\sqrt{g^{nn}}} g^{\alpha k}\partial_k g^{nn} \tag{3.13}$$

$$\Delta g_{\alpha\beta} = B_{\alpha\beta}(g, \partial g) - 2\operatorname{ric}(g)_{\alpha\beta} = B_{\alpha\beta}(g, \partial g) \tag{3.14}$$

$$g_{\alpha\beta}|_{\partial M} = h_{\alpha\beta} \tag{3.15}$$

for $1 \leq i, j \leq n$ and $1 \leq \alpha, \beta \leq n-1$. Here $B(g, \partial g)$ is a polynomial in g and its first derivatives. We remark again that these equations must be (initially) interpreted in the $L^{1,p}$ sense even though we did not write them in this way.

Let us show that (M_∞, g_∞) is a smooth Riemannian manifold with smooth metric tensor (in fact it turns out that $(M_\infty, g_\infty) = (\mathbb{R}^n_+, g_{Euc})$). As remarked above, $h_{\infty\alpha\beta} \in C^\infty$ so that [3, Proposition 5.2.2] applied to equations (3.14)-(3.15) shows that $g_{\alpha\beta} \in C^{1,\epsilon}$ for some $\epsilon > 0$. Similarly we can obtain $C^{1,\epsilon}$ regularity for the other metric components by applying [3, Proposition 5.4.1] to the equations (3.11)-(3.13). In [5] it is shown that, in harmonic coordinates, the system

$$\Phi(g) = \operatorname{ric}(g) \tag{3.16}$$

$$B_1(g) = H \tag{3.17}$$

$$B_2(g) = [h] \tag{3.18}$$

is an elliptic boundary value problem. Here $[h]$ is the pointwise conformal class of h. Since $\operatorname{ric}(g) = 0$, $H = 0$ and $[h] = [g_{Euc}]$ (and since $g_{ij} \in C^1$), elliptic regularity (see for instance [18, Theorem 6.8.3]) shows that $g_\infty \in C^\infty$ in a neighborhood of ∂M_∞. The corresponding interior estimates are well known (see for instance [4]).

Now we can show that the convergence is in the strong $L^{1,p}$ topology. Fix a term (M_i, g_i, p_i) and set $\Delta_{g_i} := \Delta_i$. Supress the subscript on g_i for the moment, setting $g := g_i$. We first look at

the tangential components $g_{\alpha\beta}$, with the dirichlet boundary condition $g_{\alpha\beta}|_{\partial M} = h_{\alpha\beta}$. We work in boundary harmonic coordinates centered at the origin, and assume that the coordinates contain a half-ball B of a definite (Euclidean) size that maps to a region in M of a definite size, independent of i. Let $B' \subset B$ be another half-ball. The estimates below are valid on B' and depend upon the distance from $\partial B'$ to ∂B. From [7] we have the estimate

$$\begin{aligned}||g_{\alpha\beta} - g_{\infty\alpha\beta}||_{L^{1,p}} \leq &C\big(\delta^2||\Delta_i g_{\alpha\beta} - \Delta_i g_{\infty\alpha\beta}||_{L^{-1,p}} + ||g_{\alpha\beta} - g_{\infty\alpha\beta}||_{L^p} \\ &+ \delta^{1/p}||h_{\alpha\beta} - h_{\infty\alpha\beta}||_{L^{1-1/p,p}}\big).\end{aligned}$$

Here δ is a fixed positive number that can be chosen independent of i. C depends on n, δ, B' and the L^∞ norm of g and therefore can be chosen independent of i. The $L^{-1,p}$ norm is the standard norm on the dual space of $L^{1,p}$. See [7] for the definition of the norm on the trace space $L^{1-1/p,p}(\mathbb{R}^{n-1})$. The strong $L^{2,p}$ convergence $h_{\alpha\beta} \to h_{\infty\alpha\beta}$ implies in particular that

$$\delta^{1/p}||h_{\alpha\beta} - h_{\infty\alpha\beta}||_{L^{1-1/p,p}} \to 0.$$

The weak $L^{1,p}$ convergence $g_{\alpha\beta} \to g_{\infty\alpha\beta}$ shows that

$$||g_{\alpha\beta} - g_{\infty\alpha\beta}||_{L^p} \to 0.$$

It remains to estimate the term

$$\begin{aligned}||\Delta_i g_{\alpha\beta} - \Delta_i g_{\infty\alpha\beta}||_{L^{-1,p}} \leq &||\Delta_i g_{\alpha\beta} - \Delta g_{\infty\alpha\beta}||_{L^{-1,p}} \\ &+ ||\Delta_i g_{\infty\alpha\beta} - \Delta g_{\infty\alpha\beta}||_{L^{-1,p}}.\end{aligned}$$

We have that $\Delta_i \to \Delta$ weakly in $L^{1,p}$ (recall that $\Delta_g = g^{ij}\frac{\partial^2}{\partial x^i \partial x^j}$ in harmonic coordinates) so that

$$||\Delta_i g_{\infty\alpha\beta} - \Delta g_{\infty\alpha\beta}||_{L^{-1,p}} \to 0.$$

Now write

$$\Delta_i g_{\alpha\beta} = B_{\alpha\beta}(g, \partial g) - 2\,\mathrm{ric}(g)_{\alpha\beta}. \tag{3.19}$$

We have that $2\,\mathrm{ric}(g_{\alpha\beta}) \to 0$ in C^0 and

$$B_{\alpha\beta}(g, \partial g) \to B_{\alpha\beta}(g_\infty, \partial g_\infty)$$

weakly in L^p. But this, of course, implies convergence in $L^{-1,p}$. Thus the tangential components of g_i converge in the strong $L^{1,p}$ topology to g_∞. For the component g^{nn} we have, also from [7], that

$$\begin{aligned}||g^{nn} - g^{nn}_\infty||_{L^{1,p}} \leq &C\big(\delta^2||\Delta_i g^{nn} - \Delta_i g^{nn}_\infty||_{L^{-1,p}} + ||g^{nn} - g^{nn}_\infty||_{L^p} \\ &+ \delta^{1/p}|| - 2(n-1)H_i g^{nn}||_{L^{-1/p,p}}\big).\end{aligned}$$

As in the previous case, the first two terms on the right side of the inequality tend to zero. We have that $g^{nn} \to g^{nn}_\infty$ in L^p and $H_i \to 0$ in C^0 so that

$$\delta^{1/p}|| - 2(n-1)H_i g^{nn}||_{L^{-1/p,p}} \to 0.$$

The components $g^{\alpha n}$ may be estimated similarly. Thus $g_i \to g_\infty$ in $L^{1,p}$.

Let us establish further regularity of the tangential components. Consider the $L^{2,p}$ estimate

$$\begin{aligned}||g_{\alpha\beta} - g_{\infty\alpha\beta}||_{L^{2,p}} \leq &C\big(\delta^2||\Delta_i g_{\alpha\beta} - \Delta_i g_{\infty\alpha\beta}||_{L^p} + ||g_{\alpha\beta} - g_{\infty\alpha\beta}||_{L^{1,p}} \\ &+ \delta^{1/p}||h_{\alpha\beta} - h_{\infty\alpha\beta}||_{L^{2-1/p,p}}\big).\end{aligned}$$

The right hand side of the estimate tends to zero since $h_{\alpha\beta} \to h_{\infty\alpha\beta}$ in $L^{2,p}$ and $g_{\alpha\beta} \to g_{\infty\alpha\beta}$ in $L^{1,p}$. This establishes a claim made in the introduction about the $L^{2,p}$ convergence of the tangential components $g_{\alpha\beta}$.

From Lemma 2.3 we see that $r_{bh}(g_\infty) = 1$. We will derive a contradiction from this by showing that $(M_\infty, g_\infty) = (\mathbb{R}^n_+, g_{Euc})$. We have already seen that the boundary $(\partial M_\infty, h_\infty) = (\mathbb{R}^{n-1}, g_{Euc})$. The fact that $i_b(M_i) \to \infty$ and the definition of pointed convergence shows that M is diffeomorphic to $\mathbb{R}^n_+$. Since M is flat, the Gauss constraint equation reads

$$||S||^2 = H^2 = 0.$$

On whatever interval boundary normal coordinates exist we have the equation

$$\nabla_{\partial t} S - S^2 = 0. \tag{3.20}$$

The initial condition $S(0) = 0$ implies that $S \equiv 0$ and thus the metric is Euclidean on this interval. On the other hand it is easy to see that boundary normal coordinates exist for all $t > 0$, since M is flat and diffeomorphic to $\mathbb{R}^n_+$. □

4. Proofs of Theorem 1.2 and Corollary 1.1

Proof of Theorem 1.2. Suppose that (M_i, g_i) is a sequence of compact, oriented, simply connected Riemannian 3-manifolds and write h_i for the induced metric on ∂M_i. To prove Theorem 1.2 it is enough to show that if

$$\begin{aligned} \mathrm{ric}(g_i) &\to 0, \\ 0 < H_i &< H_0, \\ 0 < 1/K_0 < K_i &< K_0, \\ h_i &\stackrel{GH}{\to} h \end{aligned}$$

then (M_i, g_i) subconverges to (N, g_{Euc}) in the C^α topology. Since $|\,\mathrm{ric}(g)|$ controls $|\sec(g)|$ in dimension 3, we may assume that $\sec(g_i) \to 0$. We will first show that for large i, $(M_i, g_i) \in \mathcal{M}$. If $\{e_1, e_2\}$ is an orthonormal basis so that $S(e_k) = \lambda_k e_k$ at some point p, then the Gauss equation gives

$$\lambda_1\lambda_2 = K_i - \sec_{M_i}(e_1, e_2) > 0. \tag{4.1}$$

Since $H_i = \lambda_1 + \lambda_2 > 0$ we see that λ_1 and λ_2 are both positive. The upper bound on H then implies that each λ_k is uniformly bounded above, while the inequality $K_i > 1/K_0$ shows that λ_k is uniformly bounded below. Thus there is a constant $\widetilde{H}_0 > 0$ so that

$$1/\widetilde{H}_0 < \lambda_k < \widetilde{H}_0.$$

In particular, each M_i is uniformly convex when i is large.

Let us find an upper bound for $\mathrm{diam}(M_i)$. Myers' theorem implies that the diameter of ∂M_i is bounded above. Let k be a negative lower bound for $\sec(g_i)$. If i is large enough then $-k$ can be chosen small enough so that

$$\frac{1}{\widetilde{H}_0\sqrt{-k}} > 1.$$

Thus there exists a positive solution t_0 to the equation

$$\coth(\sqrt{-k}t) = \frac{1}{\widetilde{H}_0\sqrt{-k}}$$

and by Lemma 2.1 it follows that $\mathrm{foc}(\partial M_i) \leq t_0$. To any $p \in M_i$ there exists a length minimizing geodesic from p to ∂M_i that meets ∂M_i orthogonally. Since normal geodesics do not minimize distance to the boundary past the first focal point it follows that $\mathrm{dist}(p, \partial M_i) \leq t_0$. Thus $\mathrm{diam}(M_i)$ is bounded above by $\mathrm{diam}(\partial M_i) + 2t_0$.

The Gauss-Bonnet theorem and the inequality $1/K_0 < K_i < K_0$ implies that $\mathrm{area}(\partial M_i)$ is uniformly bounded below when i is large enough.

Therefore eventually $(M_i, g_i) \in \mathcal{M}$ for appropriately defined constants. Theorem 1.1 then shows that, passing to a subsequence if necessary, (M_i, g_i) converges to an $L^{1,p}$ limit (M_∞, g_∞). Moreover, by Theorem 2.2 the metrics g_i converge in the weak $L^{2,p}_{loc}$ topology on the interior $M_\infty \backslash \partial M_\infty$. Write h_∞ for the boundary metric of g_∞. Applying Cheeger-Gromov compactness to the sequence $(\partial M_i, h_i)$ we can assume that $h_i \to h_\infty$ in the $C^{1,\alpha}$ or weak $L^{2,p}$ topology. Since $h_i \to h$ in the Gromov-Hausdorff topology we see that $(\partial M_\infty, h_\infty)$ and (Σ, h) are isometric as metric spaces. Since ∂M_∞ is orientable and admits a metric of positive curvature it follows that ∂M_∞ is diffeomorphic to S^2.

The condition $\sec(g_i) \to 0$ implies that $\sec(g_\infty) = 0$ in the $L^{2,p}$ sense in (compact regions of) the interior of M_∞. Elliptic regularity then implies that the interior of M_∞ is C^∞ and flat.

Since M_∞ is simply connected we may consider the developing map

$$\xi : M_\infty \backslash \partial M_\infty \to \mathbb{R}^3.$$

The limit M_∞ has an $L^{2,p}$ atlas of coordinate charts and therefore ξ extends to an $L^{2,p}$ (and thus $C^{1,\alpha}$) isometric immersion $M_\infty \to \mathbb{R}^3$. Let us show that the interior of $\xi(M_\infty)$ is a convex region in $\mathbb{R}^3$. Fix $q_1, q_2 \in \xi(M_\infty)$ and choose p_1, p_2 so that $\xi(p_i) = q_i$. Choose a compact set $\Omega \subset M_\infty$ so that $p_i \in \Omega$ and $\mathrm{dist}(\Omega, \partial M_\infty) = \epsilon$ for some small $\epsilon > 0$. From Theorem 2.2 it follows that $g_i \to g_\infty$ in $C^{1,\alpha}$ on Ω. Let γ_i be a geodesic with respect to the metric g_i from p_1 to p_2. We may choose ϵ small enough so that $\mathrm{Image}(\gamma_i) \subset \Omega$ for large i. The γ_i subconverge continuously (see [19]) to a g_∞-geodesic γ from p_1 to p_2. The metric g_∞ is a flat metric on a simply connected region, therefore $\xi(\gamma)$ is just the straight line segment from q_1 to q_2.

The convexity of M_∞ implies that ξ is an embedding. Composing the restriction of ξ to ∂M_∞ with a distance preserving bijection $(\partial M_\infty, h_\infty) \to (S^2, h)$, we obtain an isometric embedding (of metric spaces) from (S^2, h) to $(\mathbb{R}^3, g_{Euc})$ whose image bounds a convex connected open set.

A theorem of Pogorelov [21, Thm 3.1.6] implies that $\xi(\partial M_\infty)$ differs from N by a rigid motion of $\mathbb{R}^3$. Composing ξ by this rigid motion, we obtain the required $C^{1,\alpha}$ isometry between M_∞ and N. □

Proof of Corollary 1.1. Consider a sequence (M_i, g_i) satisfying

$$\begin{aligned} K_i &\to 1 \\ \sec(g_i) &\to 0 \\ 0 < 1/H_0 < H &< H_0. \end{aligned}$$

As demonstrated in the proof of Theorem 1.2, $M_i \in \mathcal{M}$ for large i. Thus Theorem 1.1 shows that there exists a flat limit (M_∞, g_∞) in which $g_\infty \in L^{1,p}$ on M_∞ and $g_\infty \in C^\infty$ in the interior. Since $K_i \to 1$ and ∂M_i is oriented, it follows that $\partial M_i = S^2$ and that the boundary metrics are tending to the round sphere in the $C^{1,\alpha}$ topology, any $0 < \alpha < 1$. In particular, Theorem 1.2 then shows that

the universal cover $(\overline{M}_\infty, \overline{g}_\infty)$ is isometric to the (Euclidean) unit ball in $\mathbb{R}^3$. Since the projection map is an isometry, (M_∞, g_∞) satisfies, in the $L^{1,p}$ sense, the curvature conditions

$$\begin{aligned} K &= 1, \\ \operatorname{ric}(g_\infty) &= 0, \\ S &= \operatorname{Id}. \end{aligned}$$

Similar to the analysis in Theorem 1.1, elliptic regularity implies that (M_∞, g_∞) is a smooth Riemannian manifold. Then it is easy to deduce from the above curvature conditions that (M_∞, g_∞) is isometric to the unit ball. □

5. Proof of Theorem 1.3

Proof. Let us begin by proving part i of Theorem 1.3. Suppose (M_i, g_i) is a sequence of compact, oriented Riemannian 3-manifolds with $\chi(\partial M_i) = 2$, $H_i \to 2$, and $\operatorname{ric}(g_i) \to 0$. Write h_i for the induced metrics on ∂M_i. It is sufficient to show that if $|S_i| \leq C$ and $\operatorname{diam}(M_i) \leq D$, then (M_i, g_i) subconverges to (B, g_{Euc}) in the C^α topology, where B is the unit ball. Let us show that $g_i \in \mathcal{M}$ for large enough i. As before we may assume that $\sec(g_i) \to 0$. The Gauss equation then implies that $K_i = \sec_{\partial M_i}(g_i)$ is bounded. The Gauss-Bonnet theorem

$$\int_{\partial M_i} K_i = 4\pi$$

shows that $\operatorname{area}(\partial M_i)$ is uniformly bounded below. By Theorem 1.1 it follows that (M_i, g_i) subconverges in the weak $L^{1,p}$ topology to an $L^{1,p}$ limit (M_∞, g_∞). The limit (M_∞, g_∞) satisfies weakly the equations

$$H = 2, \qquad \operatorname{ric}(g_\infty) = 0.$$

As in the proof of Theorem 1.1 we can use the fact that $h_\infty \in L^{2,p}$ and $H \in C^\infty$ to conclude that $g_\infty \in C^{1,\epsilon}$ for some $\epsilon > 0$. It follows that the developing map from the universal cover

$$\xi : \overline{M}_\infty \to \mathbb{R}^3 \tag{5.1}$$

is a $C^{2,\epsilon}$ isometric immersion. Restricting ξ to $\partial\overline{M}_\infty$ we get a $C^{2,\epsilon}$ isometric immersion of a closed 2-manifold of genus 0 and with $H = 2$ into $\mathbb{R}^3$. As mentioned in the introduction, Hopf's rigidity theorem implies that if ξ were C^3, then $\xi(\partial\overline{M}_\infty)$ is a sphere. It is straightforward to show (Lemma 5.1) that C^3 may be replaced with $C^{2,\alpha}$. Therefore $\xi(\partial\overline{M}_\infty)$ is a sphere and it follows that $K = 1$ in the $L^{2,p}$ sense. Elliptic regularity applied to the system (see the equations (3.11)-(3.15) and the arguments nearby)

$$K = 1 \quad H = 2 \quad \operatorname{ric}(g) = 0 \tag{5.2}$$

implies that $(\overline{M}_\infty, \overline{g}_\infty)$ (and thus (M_∞, g_∞)) is a smooth Riemannian 3-manifold with boundary. This, together with the equations (5.2), implies that (M_∞, g_∞) is isometric to (B, g_{Euc}).

The other cases follow from similar arguments. Let us briefly outline their proofs. Suppose that (M_i, g_i) is a sequence with $\chi(\partial M_i) = 2$, $H_i \to 2\sqrt{2}$ and $\sec(g_i) \to -1$, and suppose that $\operatorname{diam}(M_i) \leq D$ and $|S_i| \leq C$. Then Theorem 1.1 shows that $(M_i, g_i) \to (M_\infty, g_\infty)$ in the weak $L^{1,p}$ topology, where (M_∞, g_∞) is an $L^{1,p}$ Riemannian manifold satisfying

$$H = 2\sqrt{2}, \quad \sec(g_\infty) = -1.$$

Elliptic regularity then shows that $g \in C^{1,\epsilon}$ for some $\epsilon > 0$, and thus the developing map

$$\xi : \overline{M}_\infty \to \mathbb{H}^3$$

induces a $C^{2,\epsilon}$ isometric immersion of $S^2 = \partial M_\infty$ into $\mathbb{H}^3$. The proof of Lemma 5.1 shows that the second fundamental form of ξ is constant, i.e. $S = \sqrt{2}\,\mathrm{Id}$. This implies that Image(ξ) is a distance sphere in $\mathbb{H}^3$. In particular $K = 1$, where K is the Gauss curvature of g_∞. From elliptic regularity we conclude that (M_∞, g_∞) is a C^∞ Riemannian manifold, and the curvature conditions

$$K = 1, \quad H = 2\sqrt{2}, \quad \sec(g_\infty) = -1$$

imply that (M_∞, g_∞) is isometric to a metric ball in $\mathbb{H}^3$ with boundary isometric to the Euclidean sphere.

Now suppose that (M_i, g_i) is a sequence with $\sec(g_i) \to 1$, $H_i \to 0$, $\chi(\partial M_i) = 2$, $\mathrm{diam}(M_i) \leq D$ and $|S_i| \leq C$. In this case the sequence (M_i, g_i) is not contained in $\mathcal{M}$ (since $H_i \to 0$). However, if $\sec(g_i)$ is close enough to 1 and H_i is close enough to 0, then the proof of Lemma 2.2 implies that $i_b(g_i)$ is uniformly bounded below. Then the proof of Theorem 1.1 shows that the sequence (M_i, g_i) converges weakly in $L^{1,p}$ to an $L^{1,p}$ limit (M_∞, g_∞) of constant curvature 1. The developing map induces a $C^{2,\epsilon}$ minimal isometric immersion of S^2 into S^3. Almgren has shown in [1] that if this immersion were analytic, then its image would be congruent to the equator. His proof is based on the vanishing of the same holomorphic quadratic differential defined in Lemma 5.1. Therefore essentially the same proof as in Lemma 5.1 shows that 'analytic' may be replaced with '$C^{2,\epsilon}$'. This implies that M_∞ has constant Gauss curvature $K = 1$. Elliptic regularity implies that (M_∞, g_∞) is a C^∞ manifold with

$$K = 1, \quad H = 0, \quad \sec(g_\infty) = 1$$

and thus M_∞ is isometric to the upper hemisphere of (S^3, g_{+1}).

$\square$

Lemma 5.1. *Suppose (Σ, g) is a closed surface of genus 0 and $\xi : \Sigma \to \mathbb{R}^3$ is a $C^{2,\alpha}$ isometric immersion. Suppose that $H = c > 0$. Then $\xi(\Sigma)$ is a sphere.*

Proof. We will verify one of Hopf's classical proofs, applying only a minor modification. First note that g induces a complex structure on Σ, so that Σ is a Riemann surface. If u and v are isothermal coordinates on Σ, define the complex parameters

$$w = u + iv, \qquad \overline{w} = u - iv.$$

Write L, M, and N for the coefficients of the second fundamental form in the coordinates u, v and write the metric

$$g = E(du^2 + dv^2).$$

Write k_1 and k_2 for the principal curvatures. Define the complex function

$$\phi(w, \overline{w}) = \frac{L - N}{2} - iM.$$

From the formulas

$$K = k_1 k_2 = \frac{LN - M^2}{E^2}$$

and

$$H = (k_1 + k_2) = \frac{L + N}{E}$$

we see that

$$\frac{2|\phi|}{E} = |k_1 - k_2|$$

so that the zeroes of ϕ correspond to the umbilic points of Σ. Since $\xi \in C^{2,\alpha}$, we see that $E \in C^{1,\alpha}$ and L, N, $M \in C^{\alpha}$. Therefore the Codazzi equations hold weakly

$$L_v - M_u = \frac{E_v H}{2}$$
$$M_v - N_u = -\frac{E_u H}{2}$$

A simple calculation gives

$$\left(\frac{L-N}{2}\right)_u + M_v = \frac{EH_u}{2} \tag{5.3}$$
$$\left(\frac{L-N}{2}\right)_v - M_u = -\frac{EH_v}{2} \tag{5.4}$$

As H is constant, it follows that these are the Cauchy-Riemann equations for ϕ. Equations (5.3)-(5.4) form an elliptic system and from the regularity theory for weak solutions to elliptic systems (cf. [18, Thm. 6.4.4]) it follows that ϕ is complex-analytic. Therefore ϕdw^2 is a holomorphic quadratic differential. It is a consequence of Liouville's theorem that there are no nontrivial holomorphic quadratic differentials on a Riemann surface of genus 0, so that $\phi \equiv 0$. Therefore every point of M is umbillic, so that $S = k\,\mathrm{Id}$ for some $k : \Sigma \to \mathbb{R}$. The Codazzi equations show that $Dk = 0$ in the distributional sense, so that $k = \mathit{const} = c/2$. Identifying ξ with the position vector of the immersion and n for the unit normal, we get in local coordinates the system of equations

$$(n + c/2\xi)_u = 0$$
$$(n + c/2\xi)_v = 0.$$

These equations only require two derivatives of the immersion, so they are valid classically. This gives the usual proof that $\xi(M)$ is a sphere.

□

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